Spectral Methods for the Euler Equations: Part I—Fourier Methods and Shock Capturing

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Spectral methods for compressible flows are introduced in relation to finite difference and finite element techniques within the framework of the method of weighted residuals. Current spectral collocation methods are put into historical context. The basic concepts of Fourier spectral collocation methods are provided. Filtering strategies for shock-capturing approaches are also presented. Fourier shock-capturing techniques are evaluated using a one-dimensional, periodic astrophysical "nozzle" problem.

Nomenclature

a	= sound speed
N	= number of collocation points
N_f	= number of time steps between filtering
t	= physical time
u	= solution to one-dimensional test problems
$\tilde{u}$	= interpolating polynomial
u_j	= solution at collocation points
$\hat{u}_k$	= discrete Fourier coefficients
(u,v)	= physical velocities
(u_0,v_0)	= mean velocities in astrophysical problem
x_j	= collocation point
α	= coefficient for exponential cutoff
Δx	= mesh size
Δt	= time increment
ϵ	= amplitude of gravitational forcing
θ	= filter phase angle
θ_c	= filter cutoff angle
κ	= epicyclic frequency
ν	= artificial viscosity coefficient
ρ	= density
σ	= smoothing function
ϕ	= spiral phase angle
Ω	= angular frequency

Introduction

SPECTRAL methods may be viewed as an extreme development of the class of discretization schemes known by the generic name of the method of weighted residuals (MWR).[1] The key elements of the MWR are the trial functions (also called the expansion or approximating functions) and the test functions (also known as weight functions). The trial functions are used as the basis functions for a truncated series expansion of the solution that, when substituted into the differential equation, produces the residual. The test functions are used to enforce the minimization of the residual.

The choice of trial functions is what distinguishes the spectral methods from the finite element and finite difference methods. The trial functions for spectral methods are infinitely differentiable global functions. (Typically, they are tensor products of the eigenfunctions of singular Sturm-Liouville problems.) In the case of finite element methods, the domain is divided into small elements and a trial function is specified in each element. The trial functions are thus local in character and well suited for handling complex geometries. The finite difference trial functions are likewise local.

The choice of test function distinguishes between Galerkin and collocation approaches. In the Galerkin approach, the test functions are the same as the trial functions, whereas in the collocation approach the test functions are translated Dirac delta functions. In other words, the Galerkin approach is equivalent to a least-square approximation, whereas the collocation approach requires the differential equations to be satisfied exactly at the collocation points.

The collocation approach is the simplest of the MWR and appears to have been first used by Slater[2] in his study of electronic energy bands in metals. A few years later, Barta[3] applied this method to the problem of torsion in a square prism. Frazer et al.[4] developed it as a general method for solving ordinary differential equations. They used a variety of trial functions and an arbitrary distribution of collocation points. The work of Lanczos[5] established for the first time that a proper choice of the trial functions and the distribution of collocation points is crucial to the accuracy of the solution. Perhaps he should be credited with laying down the foundation of the orthogonal collocation method. This method has been revived by Clenshaw,[6] Clenshaw and Norton,[7] and Wright.[8] These studies involve application of Chebyshev polynomial expansions to initial value problems. Villadsen and Stewart[9] developed this method for boundary value problems.

The earliest applications of the spectral collocation method to initial value problems in partial differential equations were those of Kreiss and Oliger[10] (who called it the Fourier method) and Orszag[11] (who termed it pseudospectral). Details can be found in Ref. 12.

Spectral methods have been used on one-dimensional, compressible flow problems with piecewise linear solutions by Gottlieb et al.[13] and Taylor et al.[14] These reports demonstrated that spectral methods, when combined with appropriate filtering techniques, can capture one-dimensional shock waves in otherwise featureless flows. A different sort of demonstration was provided by Zang and Hussaini.[15] They exhibited spectral solutions of compressible flows with nontrivial structures in the smooth regions.

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However, a systematic assessment of the accuracy of spectral methods for compressible flows is needed to determine whether they are useful rather than merely feasible. Some comparisons with finite difference solutions were presented in Ref. 15, but the length constraint on that brief report precluded any detailed comparison. Cornille[16] presented both spectral and finite difference results for a step function solution of the inviscid Burgers' equation. His comparisons are not entirely satisfactory because the time discretization errors were not assessed and the finite difference results would surely have been better had the calculation employed a uniform grid.

The purpose of the present paper is to assess the accuracy of Fourier spectral methods when used to capture shock waves. In particular, emphasis will be placed on the role of the filtering that must be used for both stability purposes and the elimination of oscillations. One of the test problems used by Zang and Hussaini[15] provides a nontrivial test of the filtering strategies.

The restriction to Fourier methods limits the discussion here to periodic problems. However, such methods are appropriate for an important class of problems represented by flow around an airfoil. In a companion paper,[17] we address two-dimensional nonperiodic problems and the use of Chebyshev methods in conjunction with shock fitting.

Spectral Methods for Shock Capturing

Thus far, the spectral collocation method has been the only type of spectral method applied to compressible flow problems. The present discussion will be confined to spectral collocation methods, with all future references to spectral methods implicitly referring to this specific type. Since spectral methods are a novel approach to aerodynamic flow computations, a basic introduction to their properties and implementation will be presented first.

Basic Fourier Spectral Concepts

The potential accuracy of spectral methods derives from their use of suitable high-order interpolation formulas for approximating derivatives. Their efficiency has often depended on the use of fast Fourier transform techniques. An elementary example is provided by the model problem

$$u_t + u_x = 0 \tag{1}$$

with periodic boundary conditions on $[0,2\pi]$ and the initial condition

$$u(x,0) = \sin(\pi \cos x) \tag{2}$$

The standard collocation points are

$$x_j = 2\pi j/N \qquad j=0,1,\ldots,N-1 \tag{3}$$

Let u_j denote the approximation to $u(x_j)$, where the time dependence has been suppressed. Then the spatial discretization of Eq. (1) is

$$\frac{\partial u_j}{\partial t} = -\frac{\partial \tilde{u}}{\partial x}\bigg|_j \tag{4}$$

where the right-hand side is determined by first computing the discrete Fourier coefficients

$$\hat{u}_k = \frac{1}{N}\sum_{j=0}^{N-1} u_j e^{-ikx_j}, \quad k = -\frac{N}{2}, \ -\frac{N}{2}+1, \ldots, \frac{N}{2}-1 \tag{5}$$

Then the interpolating function

$$\tilde{u}(x) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k e^{ikx} \tag{6}$$

can be differentiated analytically to obtain

$$\frac{\partial \tilde{u}}{\partial x}\bigg|_j = \sum_{k=-N/2+1}^{N/2-1} ik\hat{u}_k e^{ikx_j} \tag{7}$$

(The term involving $k=-N/2$ makes a purely imaginary contribution to the sum and hence has been dropped.) Note that each derivative approximation uses all available information about the function values. The sums in Eqs. (5) and (7) can be obtained in $\mathcal{O}(N \ell n N)$ operations via the fast Fourier transform (FFT).

Table 1 Maximum error for a one-dimensional periodic problem

		Finite difference	
N	Fourier spectral	Second order	Fourth order
8	1.62 (−1)	1.11 (0)	9.62 (−1)
16	4.97 (−4)	6.13 (−1)	2.36 (1)
32	1.03 (−11)	1.99 (−1)	2.67 (−2)
64	9.55 (−12)	5.42 (−2)	1.85 (−3)
128		1.37 (−2)	1.18 (−4)

An illustration of the superior accuracy available from the spectral method for this problem is provided in Table 1. Shown there are the maximum errors at $t=1$ for the spectral method as well as for second- and fourth-order finite difference methods. The time discretization was the classical fourth-order Runge-Kutta method. In all cases, the time step was chosen so small that the temporal discretization error was negligible. Because the solution is infinitely smooth, the convergence of the spectral method on this problem is more rapid than any finite power of $1/N$. (The error for the $N=64$ spectral result is so small that it is swamped by the round-off errror of these single-precision CDC CYBER 175 calculations.) This type of convergence is usually referred to as exponential. In practical applications, the benefit of the spectral method is not the extraordinary accuracy available for large N, but rather the small size of N necessary for a moderately accurate solution.

Of course, full-scale fluid dynamics problems are considerably more complicated than those described by Eq. (1). Nevertheless, for quite a few three-dimensional, viscous, incompressible problems,[18,19] these complications have not prevented spectral methods from displaying the sort of accuracy and economy suggested by the results in Table 1.

The complications to Eq. (1) relevant to the Euler equations are: nonconstant coefficients, nonperiodic boundary conditions, a severe explicit time-step restriction, and nonlinearities causing shocks. The first three of these are also present for the incompressible Navier-Stokes equations. Nonconstant coefficient problems may be technically unstable, but this can be controlled by filtering techniques.[20] Highly accurate solutions to problems with nonperiodic boundary conditions can be obtained by using Chebyshev polynomials in place of trigonometric functions. (The use of Chebyshev polynomials is discussed in the companion paper.[17]) A widely applicable technique for surmounting the explicit time-step restriction has yet to appear. However, since this difficulty is algorithmic rather than conceptual, it should be resolved eventually. Finally, the most serious hurdle is surely the global oscillations arising from the presence of shocks in the interior of the computational domain. This also seems to be a problem in filtering, so an extended discussion of filtering in this context is provided in the next subsection.

Filtering for Fourier Spectral Methods

Several types of filtering operations are employed in spectral methods:

1) Preprocessing. The initial condition is filtered, usually in Fourier space. New grid point values are obtained from

$$u_j = \sum_{k=-N/2}^{N/2-1} \sigma(2\pi k/N)\,\hat{u}_k e^{ikx_j} \tag{8}$$

where $\sigma(\theta)$ is a non-negative function defined on $[-\pi,\pi]$ such that $\sigma(\theta)\simeq 1$ for $|\theta|\simeq 0$ and $\sigma(\theta)\to 0$ as $|\theta|\to\pi$. The coefficients $\hat{u}_k$ can be either the discrete Fourier coefficients of the original initial condition $u(x,0)$ or its continuous Fourier coefficients defined by the usual integral.

2) Derivative filtering. In the computation of spatial derivatives, the term ik in Eq. (7) is replaced with $ik\sigma(2\pi k/N)$.

3) Solution smoothing. At regular intervals in the course of advancing the solution in time, the current solution values are smoothed in Fourier space in a manner described by Eq. (8).

4) Cosmetic postprocessing. If only weak filtering is needed in order to stabilize the computations, it may be necessary to smooth the solution as above for display purposes only.

5) Artificial viscosity. Just as with finite difference computations, the equations can be modified with either a linear or nonlinear artificial viscosity.

The theoretical reasons for employing preprocessing are confined to linear problems with discontinuous solutions.[21] None of the calculations reported here employed preprocessing.

Derivative filtering has been proposed[20] as an inexpensive way to deal with the instability of Fourier spectral approximations to nonconstant coefficient problems. For a (linear) constant coefficient equation, derivative filtering is identical to a similar preprocessing and to solution smoothing. (An alternate but more costly approach is to rewrite the equations in skew-symmetric form.[22]) Our own experience with derivative filtering on more complicated problems has been unfavorable. Its absence has not led to any noticeable instability and its presence has often caused troublesome oscillations in the solution. All the results in this paper have been obtained without any sort of derivative filtering.

To date, the most successful smoothing strategy to control the oscillations that develop when shock-capturing methods are used has been solution smoothing. With this approach, however, one must choose not only the particular filter to be used, but also the frequency of application. As with artificial viscosity methods for finite difference techniques, fine tuning of the smoothing process must be done in order to obtain the best results.

Cosmetic postprocessing was introduced by Gottlieb et al.,[13] who used it on problems where only weak smoothing was needed for stability. A strong filter is then used only to make the solution presentable, not to modify the solution.

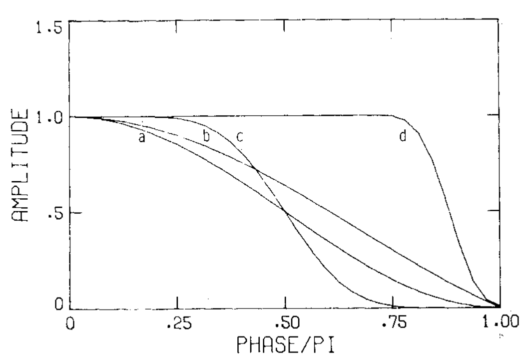

Fig. 1 Filters used in the spectral calculations. a: raised cosine; b: Lanczos; c: sharpened raised cosine; d: exponential cutoff.

Finally, an artificial viscosity can be applied, not only to stabilize the computations but to smooth them out as well. These can be viewed as nonlinear filters. They are not, however, well matched to spectral computations since they introduce a finite-order truncation error that negates the use of the high-order interpolation used by the spectral method.

Some of the smoothing functions that have been employed in our calculations are

$$\sigma(\theta) = \sin\theta/\theta \quad \text{(Lanczos)} \tag{9}$$

$$\sigma(\theta) = \tfrac{1}{2}(1+\cos\theta) \quad \text{(raised cosine)} \tag{10}$$

$$\sigma(\theta) = \sigma_0^4(35 - 84\sigma_0 + 70\sigma_0^2 - 20\sigma_0^3) \quad \text{(sharpened raised cosine)} \tag{11}$$

where σ_0 is the raised cosine given by Eq. (10),

$$\sigma(\theta) = \begin{cases} 1 & |\theta|\le\theta_c \\ \left(\dfrac{\theta-\pi}{\theta_c-\pi}\right)^4 & \theta_c\le|\theta|\le\pi \end{cases} \quad \text{(quartic taper)} \tag{12}$$

$$\sigma(\theta) = \begin{cases} 1 & |\theta|\le\theta_c \\ \exp[-\alpha(|\theta|-\theta_c)^4] & \theta_c\le|\theta|\le\pi \end{cases} \quad \text{(exponential cutoff)} \tag{13}$$

These filters are listed above in order of decreasing strength and are illustrated in Fig. 1. The Lanczos and raised cosine filters are classical. Equation (11) represents one of a number of standard formulas for sharpening a basic filter.[23] The exponential cutoff has been proposed specifically as a filter for use in spectral methods.[13,21] A popular choice for the cutoff function has been the exponential cutoff. Usually θ_c lies between $\pi/2$ and $5\pi/6$ and α is chosen so that $\sigma(\pi)$ is $\mathcal{O}(10^{-4})$ or smaller. Some successful results have been reported for this filter on linear problems.[22]

The choice of filter will determine which Fourier frequencies will be modified. The $k=0$ component of the Fourier decomposition is the only one contributing to the average value. Thus, in order to preserve the mean value of the solution to a conservation law, $\sigma(0)=1$ is required. The filters shown in Fig. 1 all have this property. However, the effect of the filter on the high frequencies is usually more difficult to assess in nonlinear problems.

The raised cosine (also known as the von Hann window) admits a simple physical interpretation. It is algebraically equivalent to

$$u_j - \frac{u_{j-1}+2u_j+u_{j+1}}{4} = u_j + \frac{(\Delta x)^2}{4}\,\frac{u_{j-1}-2u_j+u_{j+1}}{(\Delta x)^2} \tag{14}$$

This is clearly a second-order artificial viscosity term with the coefficient

$$\frac{(\Delta x)^2}{4(\Delta t/N_f)} \tag{15}$$

where N_f is the number of time steps between applications of the filter. Figure 1 suggests that the other types of filters amount to nonphysical viscosities: they damp preferentially the high-frequency components of the solution, but in a different manner than a physical viscosity.

The frequency of applying the filter is analogous to the selection of the size of the artificial viscosity for finite difference methods. Applied too often, a strong filter such as the Lanczos will unacceptably smear out a shock. On the other hand, frequent applications of a weak filter such as the ex-

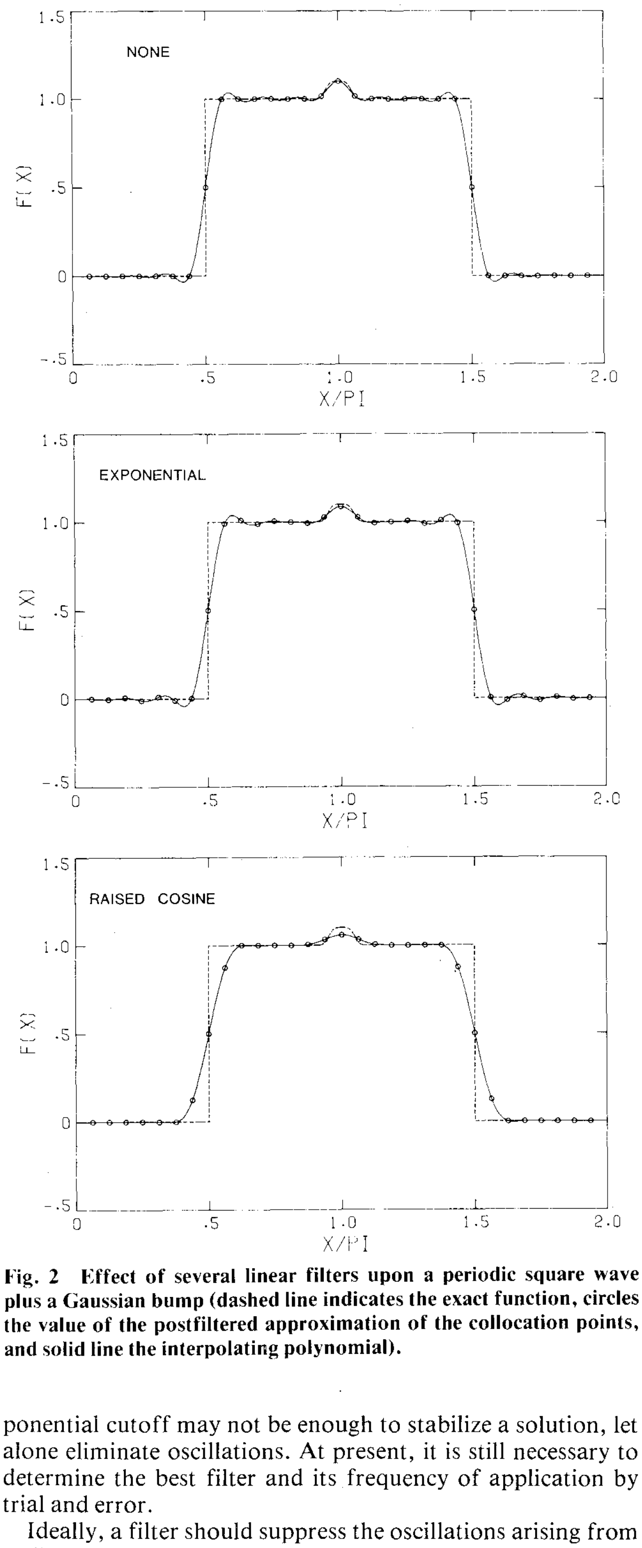

Fig. 2 Effect of several linear filters upon a periodic square wave plus a Gaussian bump (dashed line indicates the exact function, circles the value of the postfiltered approximation of the collocation points, and solid line the interpolating polynomial).

ponential cutoff may not be enough to stabilize a solution, let alone eliminate oscillations. At present, it is still necessary to determine the best filter and its frequency of application by trial and error.

Ideally, a filter should suppress the oscillations arising from a discontinuity while retaining the accuracy of the spectral method in the structured regions of the flow. None of the filters described above is entirely satisfactory. A simple illustration is furnished in Fig. 2. Shown there are several approximations to one period of a function composed of a square wave plus a small, narrow Gaussian. The approximations are based on the interpolating polynomials generated by Eqs. (5) and (6) with $N=32$. For the bottom two parts of Fig. 2, the Fourier coefficients have been modified by the raised cosine and the exponential (with $\theta_c=2\pi/3$) filters, respectively. The oscillations of the interpolating polynomials between the grid points are significant for real problems because they are indicative of the dangerous high-frequency modes that will interact with variable-coefficient and nonlinear terms.

The familiar dilemma illustrated by Fig. 2 led to the adoption of nonlinear filters in central difference schemes. As with artificial viscosities, the nonlinear filters currently available introduce a truncation error of finite order into the overall scheme, apparently precluding them from converging exponentially fast even in the smooth regions of the flow. These considerations imply that spectral shock-capturing solutions of compressible flows must be examined critically.

Results for Fourier Spectral Shock Capturing

Discussions of shock-capturing techniques are easiest for the Fourier spectral methods. The discrete operator is simple, the collocation points are uniformly distributed, and the boundary conditions pose no difficulty. However, there does not appear to be any interesting one-dimensional aerodynamic problem with periodic boundary conditions. One nontrivial test case for spectral shock-capturing techniques uses an approximate set of equations derived by Woodward[24] for studying the time development of shock waves in a spiral galaxy. The equations describing an isothermal gaseous component in a very thin disk galaxy are

$$\rho_t + (\rho u)_\phi = 0 \tag{16a}$$

$$(\rho u)_t + [\rho(u^2+a^2)]_\phi = 2\Omega(v-v_0)\rho + \rho\epsilon\sin\phi \tag{16b}$$

$$(\rho v)_t + (\rho uv)_\phi = -(\kappa^2/2\Omega)(u-u_0)\rho \tag{16c}$$

The boundary conditions are periodic in ϕ with period 2π. The last term in Eq. (16b) represents the gravitational forcing of the gas from the spiral field of the much more massive stellar component. These equations have some significant differences from the Euler equations, notably the forcing term and the asymmetrical role of the velocity components. A more detailed explanation of this physical problem and the approximations used in deriving Eqs. (16) is available in Ref. 24.

The parameter ϵ in Eq. (16b) is a dimensionless measure of the strength of the gravitational forcing. If this forcing is sufficiently strong, then the steady-state solution to this astrophysical problem contains a shock. Behind the shock there is a region of rapid decompression and further downstream occur some features due to the second harmonic of the forcing term. The steady-state solution, then, is more complex than some standard test problems (such as the one-dimensional shock tube) whose solutions are merely piecewise linear functions. The challenge for the spectral method is to capture the shock and to suppress its attendant oscillations without also destroying the remaining structure of the solution.

The specific test problem uses (the units are not of interest here) $a=8.56$, $\Omega=21.37$, $v_0=115$, $\epsilon=72.92$, $\kappa=26.75$, and $u_0=13.5$.

The spectral calculations use the steady-state solution to Eqs. (16) as the initial condition. The transients that would be generated by some other initial condition take a very long time to damp out because spectral methods have very low inherent damping and there are no boundaries out of which transients can convect. The temporal discretization uses a second-order Adams-Bashforth predictor followed by a third-order Adams-Moulton corrector. It has a CFL limit of $3/(2\pi)$. The calculations used a CFL number of 0.2 and were run for 500 time steps. We seek to illustrate the performance of filtering procedures ranging from the very weak to the very strong. The specific examples to follow are representative of many dozens of calculations in which the type of filtering, the form of the filtering function, and the frequency of filtering were each varied over a wide range.

Figure 3 shows the effect of applying the weak exponential cutoff filter (with $\theta_c=0.7\pi$ and $\alpha=5$) every 50 time steps. Both high- and low-frequency oscillations are quite evident. If

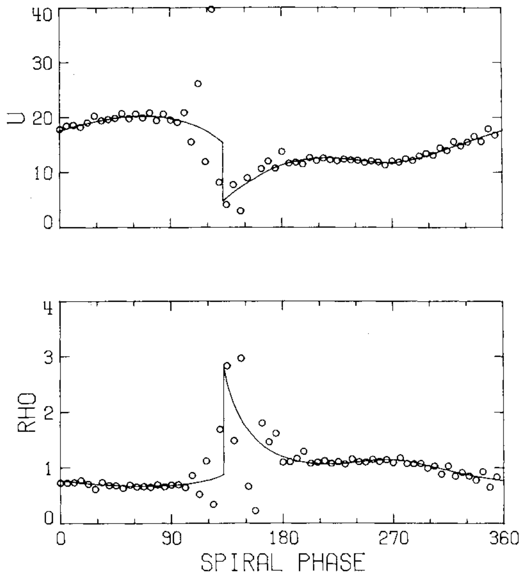

Fig. 3 Spectral solution to the astrophysical model (circles) computed with the exponential cutoff filter applied every 50 steps (solid line is the exact solution).

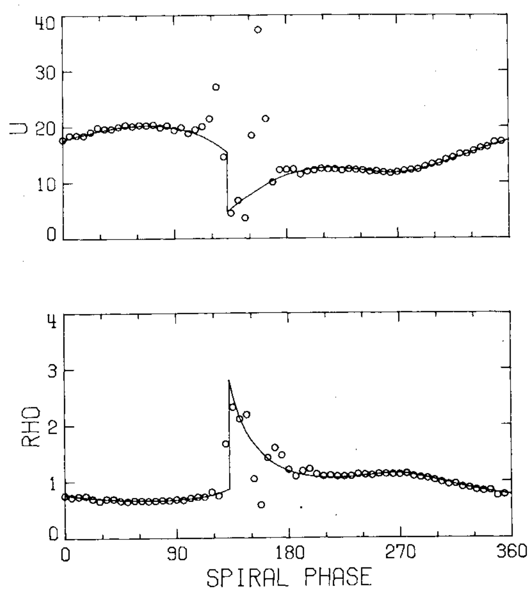

Fig. 4 Solution as in Fig. 2, but with a Lanczos cosmetic filter also applied.

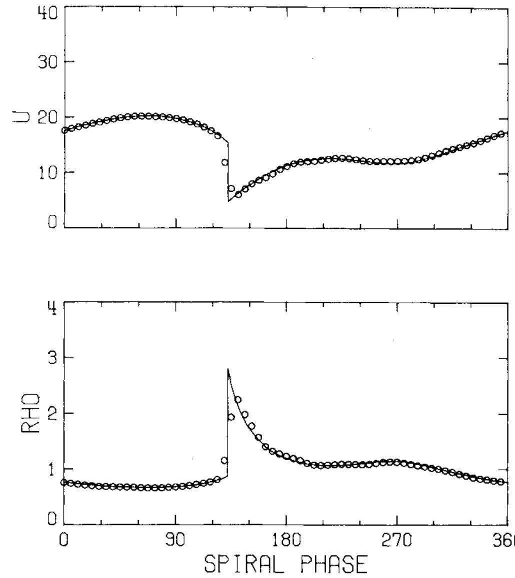

Fig. 5 Spectral solution to the astrophysical model with a raised cosine filter applied every 50 steps.

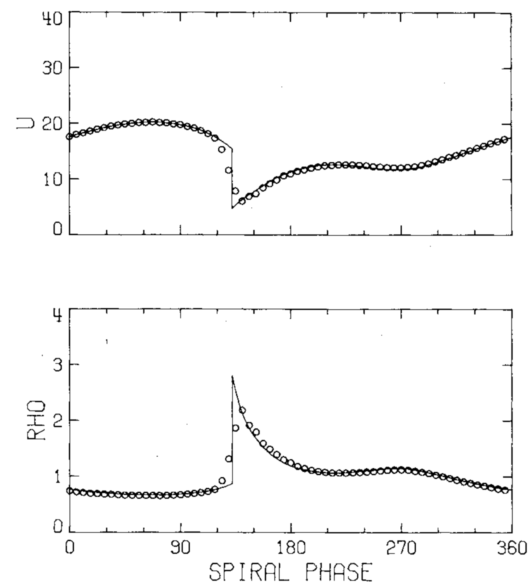

Fig. 6 Spectral solution to the astrophysical model with a nonlinear artificial viscosity.

a Lanczos cosmetic postprocessing step is added, then the results shown in Fig. 4 are obtained. As one would expect, the high-frequency oscillations are under control, but the low-frequency errors remain substantial. The failure of the weak filtering strategy on this problem (even when applied every time step) may appear puzzling in view of the success a similar strategy has achieved on a standard shock-tube problem.[13] We attribute the difference to the strong expansion in the postshock region of the present problem. Without appreciably stronger smoothing, something resembling an expansion shock will form in this region. The dip in the density plot is not smoothed by a weak filter and it grows until the density becomes negative and the calculation breaks down. On the other hand, if the forcing parameter ϵ is reduced to 31.0, then the weak exponential filter is sufficient to produce a stable computation. However, the shock here is quite weak (the density ratio is 1.38). The more difficult problem for $\epsilon = 72.92$ evidently requires more drastic filtering.

Table 2 Comparison of L_2 errors for filtering and artificial viscosity

N	Filtering	Artificial viscosity
16	0.087	0.098
32	0.036	0.037
64	0.016	0.017

Stable computations are obtained by applying the raised cosine filter every 50 time steps. As Fig. 5 indicates, the expansion is now adequately controlled and the shock is captured in two points. But notice that there are some low-frequency errors in the vicinity of the second harmonic near a spiral phase of 270 deg.

An alternative form of strong filtering is to add a nonlinear artificial viscosity. In his second-order MacCormack's method calculations for the two-dimensional version of this problem, Liebovitch[25] used a second-order viscosity. In the

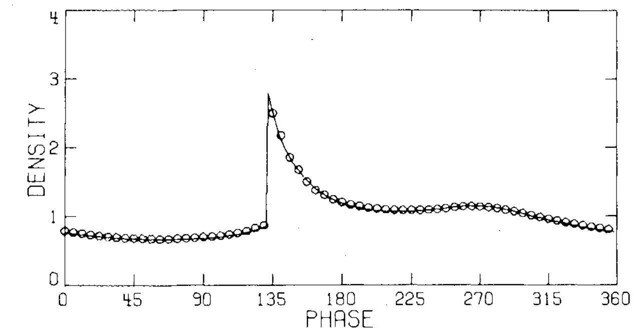

Fig. 7 Spectral finite difference solution using a weak artifical viscosity and raised cosine filter (one-sided linear extrapolation is used at the shock when smoothing is applied).

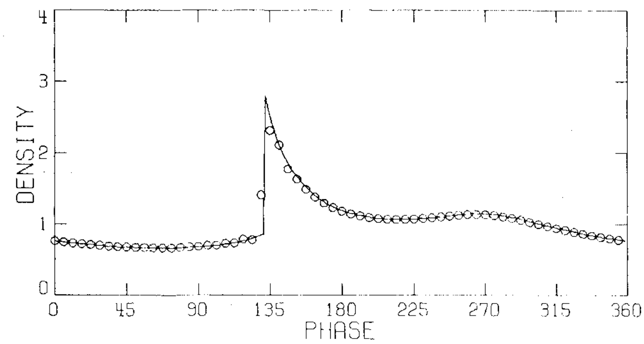

Fig. 8 MacCormack solution to the astrophysical model.

density equation this was proportional to

$$\nu_{i+1/2}(\rho_{i+1}-\rho_i)-\nu_{i-1/2}(\rho_i-\rho_{i-1}) \tag{17}$$

where

$$\nu_{i+1/2}=|u_{i+1}-u_i| \tag{18}$$

Similar terms were used for the two momentum equations with the appropriate momentum variable replacing the density variable in Eq. (17). Figure 6 shows the results of employing this nonlinear artificial viscosity instead of any type of linear filtering. Compared with Fig. 5, there is now one more point in the shock but the solution is smoother, particularly near the second harmonic. A cosmetic post-processing step that is one-sided near the shock would certainly sharpen the latter solution.

The need for strong filtering to keep the computations stable limits the accuracy obtained for this problem. Table 2 shows the discrete L_2 errors in the density, excluding the shock region for both the strongly filtered and artificial viscosity solutions on grids of 16, 32, and 64 points. Away from the shock, both approaches give very similar errors and both show first-order error decay rates. Nothing approaching "spectral accuracy," i.e., exponential convergence, has been obtained for this problem.

A comparison with the preliminary results[15] for this problem is in order. The earlier spectral result is given here in Fig. 7. The spectral method used a second-order linear artificial viscosity applied every time step, with the raised cosine solution smoothing added every 100 time steps in physical space [see Eq. (14)], but modified for one-sided linear extrapolation on the two points straddling the shock. This latter feature accounts for the apparent sharpness of the shock. Unfortunately, that type of one-sided averaging was not conservative. Inspection of Fig. 7 reveals that the total mass has increased. Conservative versions of this one-sided averaging have not been successful in producing sharp shocks.

All three spectral results may be compared with the second-order MacCormack's solution displayed in Fig. 8. A nonlinear artificial viscosity as given by Eq. (17) was included in these calculations. Grid refinement studies for this method also reveal essentially first-order error decay. Away from the shock, this straightforward finite difference result appears to be superior to all of the spectral results. Upwind difference schemes can do even better in resolving the shock than MacCormack's method. Results for a second-order flux splitting method applied to this problem are given in Refs. 26 and 27. In terms of the error norm used in Ref. 26, the error in the density is 1.6 and 2.4% for the present spectral filtering and artificial viscosity calculations, respectively. For Van Leer's second-order scheme, it is only 0.6%.

Conclusions

The astrophysical problem is the most challenging one-dimensional compressible flow problem for which spectral shock-capturing results have been reported. Cornille[16] examined a scalar problem for which the MacCormack finite difference method had sufficient implicit dissipation. In contrast, explicit dissipation had to be added to this finite difference method for it to handle the astrophysical problem. Cornille's spectral methods evidently required no explicit filtering on his problem because the shock is relatively weak and because they have an implicit dissipation similar to that of MacCormack's method. The shock-tube problem of Gottlieb et al.[13] required only a weak exponential filter, whereas much stronger linear filtering was required for the astrophysical problem. The numerical examples of Taylor et al.[14] were simpler than the shock-tube problem, consisting of only a single wave. The piecewise linear nature of these other test problems precludes their use for assessing the accuracy of a numerical method.

The performance of the spectral method on the astrophysical problem should serve as a caution to those considering this approach. The only presentable solutions we obtained employed such drastic filtering that the accuracy of the method deteriorated to first order. In a direct comparison, finite difference methods produced better (and cheaper) solutions.

We are not advocating that spectral methods be abandoned for compressible flows. There are circumstances under which they are quite competitive. Streett[28] has developed a spectral shock-capturing method for the two-dimensional full potential equation. His spectral method requires far fewer grid points than the best finite difference methods to achieve engineering accuracy for supercritical, lifting airfoils. Moreover, the application of spectral multigrid techniques[29] makes the cost of this method comparable to that of finite difference methods. Streett's results are so accurate, albeit not exponentially so, because only a weak filter is required. The Gibbs oscillations are less severe for this problem than for the astrophysical problem because the solution itself is continuous at the shock; it is the derivative that is discontinuous.

Exponentially convergent spectral solutions to the Euler equations can be obtained by resorting to shock-fitting techniques. These methods are described for two-dimensional flows in the sequel to the present paper.[17]

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